

## Chapter 1

### Lexicographic Composition of Similarity-Based Fuzzy Orderings

Ulrich Bodenhofer

*Institute of Bioinformatics, Johannes Kepler University Linz  
Altenberger Str. 69, A-4040 Linz, Austria  
bodenhofer@bioinf.jku.at*

The present paper introduces an approach to construct lexicographic compositions of similarity-based fuzzy orderings. This construction is demonstrated by means of non-trivial examples. As this is a crucial feature of lexicographic composition, the preservation of linearity is studied in detail. We obtain once again that it is essential for meaningful results that the t-norm under consideration induces an involutive (strong) negation.

#### 1.1. Introduction

Lexicographic composition is a fundamental construction principle for ordering relations. The most important feature of this construction is that the composition of two linear orderings again yields a linear ordering. Given two orderings  $\leq_1$  and  $\leq_2$  on non-empty domains  $X_1$  and  $X_2$ , respectively, the lexicographic composition is an ordering  $\leq'$  on the Cartesian product  $X_1 \times X_2$ , where  $(x_1, x_2) \leq' (y_1, y_2)$  if and only if

$$(x_1 \neq y_1 \wedge x_1 \leq_1 y_1) \vee (x_1 = y_1 \wedge x_2 \leq_2 y_2). \quad (1.1)$$

Rewriting  $x_1 \neq y_1 \wedge x_1 \leq_1 y_1$  as  $x_1 <_1 y_1$  (i.e. the strict ordering induced by  $\leq_1$ ) and taking into account that  $x_1 = y_1 \vee x_1 \neq y_1$  is a tautology and that  $\leq_1$  is reflexive, we obtain that (1.1) is equivalent to

$$(x_1 \leq_1 y_1 \wedge x_2 \leq_2 y_2) \vee x_1 <_1 y_1. \quad (1.2)$$

The study of fuzzy orderings can be traced back to the early days of fuzzy set theory.<sup>1-4</sup> Partial fuzzy orderings in the sense of Zadeh,<sup>1</sup> however, have severe shortcomings that were finally resolved by replacing the

crisp equality by a fuzzy equivalence relation, thereby maintaining the well-known classical fact that orderings are obtained from preorderings by factorization.<sup>5-9</sup>

In previous works of the author,<sup>7,10</sup> several methods for constructing fuzzy orderings are presented, including Cartesian products. How to transfer lexicographic composition to the fuzzy framework, however, remained an open problem. The reason why this remained an open issue for a relatively long time is that there was no meaningful concept of strict fuzzy ordering in the similarity-based framework so far. As this issue is solved now,<sup>11</sup> we are able to give a solution in this paper. For proof details, the reader is referred to upcoming publications.

## 1.2. Preliminaries

For simplicity, we consider the unit interval  $[0, 1]$  as our domain of truth values in this paper. Note that most results, with only minor and obvious modifications, also hold for more general structures.<sup>5,9,12,13</sup> The symbols  $T$ ,  $\tilde{T}$ , etc., denote left-continuous t-norms.<sup>14</sup> Correspondingly,  $\bar{T}$  denotes the unique residual implication of  $T$ . Furthermore, we denote the residual negation of  $T$  with  $N_T(x) = \bar{T}(x, 0)$ .

**Definition 1.1.** A binary fuzzy relation  $E : X^2 \rightarrow [0, 1]$  is called *fuzzy equivalence relation*<sup>a</sup> with respect to  $T$ , for brevity *T-equivalence*, if the following three axioms are fulfilled for all  $x, y, z \in X$ :

- (1) Reflexivity:  $E(x, x) = 1$
- (2) Symmetry:  $E(x, y) = E(y, x)$
- (3)  $T$ -transitivity:  $T(E(x, y), E(y, z)) \leq E(x, z)$

**Definition 1.2.** A binary fuzzy relation  $L : X^2 \rightarrow [0, 1]$  is called *fuzzy ordering* with respect to  $T$  and a  $T$ -equivalence  $E : X^2 \rightarrow [0, 1]$ , for brevity *T-E-ordering*, if it fulfills the following three axioms for all  $x, y, z \in X$ :

- (1)  $E$ -reflexivity:  $E(x, y) \leq L(x, y)$
- (2)  $T$ - $E$ -antisymmetry:  $T(L(x, y), L(y, x)) \leq E(x, y)$
- (3)  $T$ -transitivity:  $T(L(x, y), L(y, z)) \leq L(x, z)$

<sup>a</sup>Note that various diverging names for this class of fuzzy relations appear in literature, like similarity relations, indistinguishability operators, equality relations, and several more.<sup>1,13,15,16</sup>

**Definition 1.3.** A fuzzy relation  $R : X^2 \rightarrow [0, 1]$  is called *strongly complete* if  $\max(L(x, y), L(y, x)) = 1$  for all  $x, y \in X$ .<sup>2,4,17</sup>  $R$  is called *T-linear* if  $N_T(L(x, y)) \leq L(y, x)$  for all  $x, y \in X$ .<sup>5,17</sup>

**Definition 1.4.** A binary fuzzy relation  $S : X^2 \rightarrow [0, 1]$  is called *strict fuzzy ordering* with respect to  $T$  and a  $T$ -equivalence  $E : X^2 \rightarrow [0, 1]$ , for brevity *strict T-E-ordering*, if it fulfills the following axioms for all  $x, x', y, y', z \in X$ :

- (1) Irreflexivity:  $S(x, x) = 0$
- (2)  $T$ -transitivity:  $T(S(x, y), S(y, z)) \leq S(x, z)$
- (3)  $E$ -extensionality:  $T(E(x, x'), E(y, y'), S(x, y)) \leq S(x', y')$

As already mentioned above, it is of vital importance for lexicographic composition how to “strictify” a given fuzzy ordering. The following theorem summarizes the most important facts.

**Theorem 1.1 (Bodenhofer & Demirci, 2005<sup>11</sup>).** Consider a  $T$ -equivalence  $E : X^2 \rightarrow [0, 1]$  and a  $T$ - $E$ -ordering  $L : X^2 \rightarrow [0, 1]$ . Then the following fuzzy relation is a strict  $T$ - $E$ -ordering on  $X$ :

$$S(x, y) = \min(L(x, y), N_T(L(y, x)))$$

Moreover,  $S$  is monotonic with respect to  $L$  in the following sense (for all  $x, y, z \in X$ ).

$$\begin{aligned} T(L(x, y), S(y, z)) &\leq S(x, z) \\ T(S(x, y), L(y, z)) &\leq S(x, z) \end{aligned}$$

$S$  is the largest strict  $T$ - $E$ -ordering contained in  $L$  that fulfills this kind of monotonicity.

For intersecting  $T$ -transitive fuzzy relations, the concept of dominance of t-norms is of vital importance.<sup>14,18,19</sup>

**Definition 1.5.** A t-norm  $T_1$  is said to *dominate* another t-norm  $T_2$  if, for every quadruple  $(x, y, u, v) \in [0, 1]^4$ , the following holds:

$$T_1(T_2(x, y), T_2(u, v)) \geq T_2(T_1(x, u), T_1(y, v))$$

**Lemma 1.1 (De Baets & Mesiar, 1998<sup>18</sup>).** Consider two t-norms  $T_1$  and  $T_2$ . The  $T_1$ -intersection of any two arbitrary  $T_2$ -transitive fuzzy relations is  $T_2$ -transitive if and only if  $T_1$  dominates  $T_2$ .

### 1.3. Starting the Easy Way: One Crisp and One Fuzzy Ordering

Let us first consider the case where the primary ordering is crisp and the secondary ordering is fuzzy. As the strict ordering is only needed for the primary ordering, we do not need to take any strict fuzzy ordering into account.

**Proposition 1.1.** *Let us consider a crisp ordering  $L_1 : X_1^2 \rightarrow \{0, 1\}$ , a  $T$ -equivalence  $E_2 : X_2^2 \rightarrow [0, 1]$ , and a  $T$ - $E_2$ -ordering  $L_2 : X_2^2 \rightarrow [0, 1]$ . Then the fuzzy relation  $L : (X_1 \times X_2)^2 \rightarrow [0, 1]$  defined as*

$$L((x_1, x_2), (y_1, y_2)) = \begin{cases} 1 & \text{if } x_1 \neq y_1 \text{ and } L_1(x_1, y_1) = 1, \\ L_2(x_2, y_2) & \text{if } x_1 = y_1, \\ 0 & \text{otherwise,} \end{cases}$$

*is a fuzzy ordering with respect to  $T$  and the  $T$ -equivalence  $E : (X_1 \times X_2)^2 \rightarrow [0, 1]$  defined as*

$$E((x_1, x_2), (y_1, y_2)) = \begin{cases} E_2(x_2, y_2) & \text{if } x_1 = y_1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, if both components  $L_1$  and  $L_2$  are crisp orderings, then  $L$  as defined above is equivalent to the constructions (1.1) and (1.2). Moreover,  $E$  as defined above is nothing else but the Cartesian product of the crisp equality with  $E_2$ .

**Example 1.1.** Consider  $X_1 = X_2 = [0, 4]$ , let  $L_1$  be the classical linear ordering of real numbers, and assume that  $L_2$  is defined as follows:

$$L_2(x, y) = \max(\min(1 - x + y, 1), 0)$$

It is easy to see that  $L_2$  is a strongly complete fuzzy ordering with respect to the Łukasiewicz t-norm  $T_L(x, y) = \max(x + y - 1, 0)$  and the  $T_L$ -equivalence  $E_2(x, y) = \max(1 - |x - y|, 0)$ . Figure 1.1 shows a cut view of the fuzzy ordering  $L$  that is obtained when applying the construction from Proposition 1.1. The cut view has been obtained by plotting the value  $L((2, 2), (y_1, y_2))$  as a two-dimensional function of  $y_1$  and  $y_2$ .

The following proposition clarifies in which way linearity of the two component orderings  $L_1$  and  $L_2$  is preserved by the construction in the previous proposition.

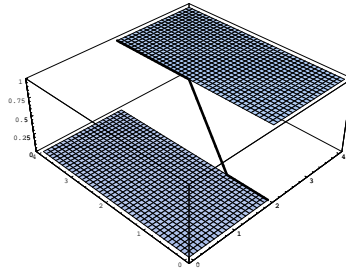


Fig. 1.1. Cut view of a lexicographic composition of a crisp linear ordering and a fuzzy ordering according to Proposition 1.1.

**Proposition 1.2.** *Let us make the same assumptions as in Proposition 1.1. If  $L_1$  is a crisp linear ordering and  $L_2$  is strongly complete, then  $L$  is also strongly complete. If  $L_1$  is a crisp linear ordering and  $L_2$  is  $T$ -linear, then  $L$  is also  $T$ -linear.*

#### 1.4. Lexicographic Composition of Two Non-Trivial Fuzzy Orderings

The results of the previous section have been known to the author since 1998, but they were not published so far, as they cannot be considered a full-fledged solution of the problem. So let us now consider the general case, where both components are fuzzy orderings without any further assumptions. The following theorem gives a general construction inspired by the classical construction (1.2).

**Theorem 1.2.** *Consider two  $T$ -equivalences  $E_1 : X_1^2 \rightarrow [0, 1]$ ,  $E_2 : X_2^2 \rightarrow [0, 1]$ , a  $T$ - $E_1$ -ordering  $L_1 : X_1^2 \rightarrow [0, 1]$ , and a  $T$ - $E_2$ -ordering  $L_2 : X_2^2 \rightarrow [0, 1]$ . Moreover, let  $\tilde{T}$  be a  $t$ -norm that dominates  $T$ . Then the fuzzy relation  $\text{Lex}_{\tilde{T}, T}(L_1, L_2) : (X_1 \times X_2)^2 \rightarrow [0, 1]$  defined as*

$$\begin{aligned} \text{Lex}_{\tilde{T}, T}(L_1, L_2)((x_1, x_2), (y_1, y_2)) = \\ \max(\tilde{T}(L_1(x_1, y_1), L_2(x_2, y_2)), \\ \min(L_1(x_1, y_1), N_T(L_1(y_1, x_1)))) \end{aligned} \tag{1.3}$$

*is a fuzzy ordering with respect to  $T$  and the  $T$ -equivalence  $\text{Cart}_{\tilde{T}}(E_1, E_2) : (X_1 \times X_2)^2 \rightarrow [0, 1]$  defined as the Cartesian product of  $E_1$  and  $E_2$ :*

$$\text{Cart}_{\tilde{T}}(E_1, E_2)((x_1, x_2), (y_1, y_2)) = \tilde{T}(E_1(x_1, y_1), E_2(x_2, y_2))$$

Note that, if  $L_1$  is a crisp ordering, then  $\text{Lex}_{\bar{T},T}(L_1, L_2)$  defined as in Theorem 1.2 coincides with the fuzzy relation  $L$  defined in Proposition 1.1. Consequently, if both components  $L_1$  and  $L_2$  are crisp orderings, then  $\text{Lex}_{\bar{T},T}(L_1, L_2)$  is equivalent to the constructions (1.1) and (1.2).

Construction (1.3) is based on one specific formulation of lexicographic composition, namely (1.2). This is just one possible way of defining lexicographic composition. It is unknown whether there are other meaningful ways to define lexicographic composition on the basis of a different formulation that is equivalent to (1.2) in the classical Boolean case.

**Example 1.2.** Consider again the domain  $X = [0, 4]$  and consider the following three fuzzy relations on  $X$ :

$$\begin{aligned} L_3(x, y) &= \max(\min(1 - \frac{1}{2}(x - y), 1), 0) \\ L_4(x, y) &= \min(\exp(y - x), 1) \\ L_5(x, y) &= \min(\exp(3(y - x)), 1) \end{aligned}$$

$L_3$  is a  $T_{\mathbf{L}}$ - $E_3$ -ordering with  $E_3(x, y) = \max(1 - \frac{1}{2}|x - y|, 0)$ .  $L_4$  is a  $T_{\mathbf{P}}$ - $E_4$ -ordering<sup>b</sup> with  $E_4(x, y) = \exp(-|x - y|)$  and, since  $T_{\mathbf{L}} \leq T_{\mathbf{P}}$ , a  $T_{\mathbf{L}}$ - $E_4$ -ordering as well.  $L_5$  is a  $T_{\mathbf{P}}$ - $E_5$ -ordering with  $E_5(x, y) = \exp(-3|x - y|)$  and a  $T_{\mathbf{L}}$ - $E_5$ -ordering as well. Thus we can define the following fuzzy relations from the fuzzy orderings  $L_2$  (from Example 1.1),  $L_3$ ,  $L_4$ , and  $L_5$ :

$$\begin{aligned} \mathbf{L}_a &= \text{Lex}_{T_{\mathbf{M}},T_{\mathbf{L}}}(L_2, L_2) \\ \mathbf{L}_b &= \text{Lex}_{T_{\mathbf{L}},T_{\mathbf{L}}}(L_3, L_2) \\ \mathbf{L}_c &= \text{Lex}_{T_{\mathbf{P}},T_{\mathbf{L}}}(L_4, L_2) \\ \mathbf{L}_d &= \text{Lex}_{T_{\mathbf{P}},T_{\mathbf{L}}}(L_5, L_2) \end{aligned}$$

Theorem 1.2 then ensures that all these four fuzzy relations are fuzzy orderings with respect to the Lukasiewicz t-norm  $T_{\mathbf{L}}$  and  $T_{\mathbf{L}}$ -equivalences defined as the corresponding Cartesian products. Figure 1.2 shows cut views of the four lexicographic compositions, where we keep the first argument vector constant (we choose  $(x_1, x_2) = (2, 2)$ ) and plot the value  $\mathbf{L}_*((2, 2), (y_1, y_2))$  as a two-dimensional function of  $y_1$  and  $y_2$ .

Now the question arises whether the lexicographic composition of two linear fuzzy orderings is again linear. Note that there are several notions of linearity of fuzzy orderings.<sup>17</sup> Let us first consider strong completeness.

<sup>b</sup>Here  $T_{\mathbf{P}}$  denotes the product t-norm; note that  $T_{\mathbf{P}}$  dominates  $T_{\mathbf{L}}$ .<sup>19</sup>

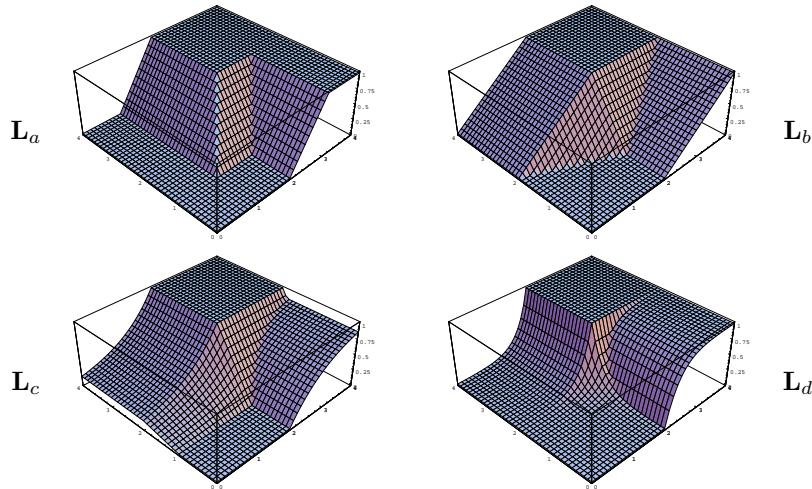


Fig. 1.2. Cut views of the four lexicographic compositions from Example 1.2.

**Example 1.3.** All fuzzy orderings considered in Examples 1.1 and 1.2 are strongly complete. Note, however, that none of the lexicographic compositions defined in Example 1.2 is strongly complete. To demonstrate that, consider the plots in Figure 1.3. These two plots show the values

$$\begin{aligned} & \max(\mathbf{L}_a((2, 2), (y_1, y_2)), \mathbf{L}_a((y_1, y_2), (2, 2))) \\ & \max(\mathbf{L}_d((2, 2), (y_1, y_2)), \mathbf{L}_d((y_1, y_2), (2, 2))) \end{aligned}$$

as two-dimensional functions of  $y_1$  and  $y_2$ . If  $\mathbf{L}_a$  and  $\mathbf{L}_d$  were strongly complete, these two functions would have to be the constant 1, which is obviously not the case. The same is true for the two other lexicographic compositions  $\mathbf{L}_b$  and  $\mathbf{L}_c$ .

After this negative answer, let us relax the question a bit and attempt the question whether the lexicographic composition of two strongly complete fuzzy orderings is  $T$ -linear.

**Proposition 1.3.** *Let us make the same assumptions as for Theorem 1.2. If  $L_1$  and  $L_2$  are strongly complete fuzzy orderings and the residual negation  $N_T$  is involutive (i.e.  $N_T(N_T(x)) = x$  holds for all  $x \in [0, 1]$ ), then the fuzzy ordering  $\text{Lex}_{\tilde{T}, T}(L_1, L_2)$  is  $T$ -linear.*

Note that Proposition 1.3 also proves that all the four lexicographic compositions defined in Example 1.2 are  $T_L$ -linear.

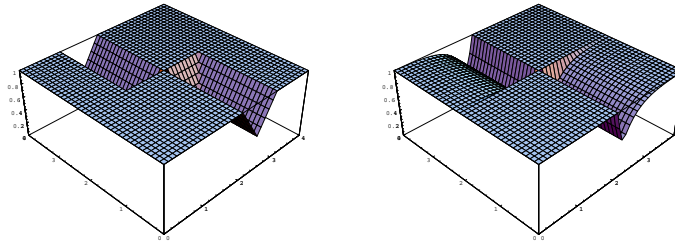


Fig. 1.3. Plots of the functions  $\max(\mathbf{L}_a((2, 2), (y_1, y_2)), \mathbf{L}_a((y_1, y_2), (2, 2)))$  (left) and  $\max(\mathbf{L}_d((2, 2), (y_1, y_2)), \mathbf{L}_d((y_1, y_2), (2, 2)))$  (right).

The proof of Proposition 1.3 does not work if we do not assume that  $N_T$  is an involution. The question arises, of course, whether this condition is not only sufficient, but also necessary. The answer is that this is the case, as the following example demonstrates.

**Example 1.4.** Consider a left-continuous t-norm for which a value  $z \in ]0, 1[$  exists such that  $N_T(N_T(z)) \neq z$ . Since  $N_T(N_T(z)) \geq z$  always holds, we can infer, in this case, that  $N_T(N_T(z)) > z$  must hold. Now let us consider two simple strongly complete fuzzy orderings on the sets  $X_1 = \{a, b\}$  and  $X_2 = \{c, d\}$ , respectively:

$$\begin{array}{c|cc} L_6 & a & b \\ \hline a & 1 & 1 \\ b & z & 1 \end{array} \quad \begin{array}{c|cc} L_7 & c & d \\ \hline c & 1 & 1 \\ d & 0 & 1 \end{array}$$

Then we can infer the following for any choice of  $\tilde{T}$ :

$$\begin{aligned} \text{Lex}_{\tilde{T}, T}(L_6, L_7)((a, d), (b, c)) &= N_T(L_6(b, a)) = N_T(z) \\ \text{Lex}_{\tilde{T}, T}(L_6, L_7)((b, c), (a, d)) &= L_6(b, a) = z \end{aligned}$$

Hence we obtain that

$$\begin{aligned} N_T(\text{Lex}_{\tilde{T}, T}(L_6, L_7)((a, d), (b, c))) &= N_T(N_T(z)) \\ &> z = \text{Lex}_{\tilde{T}, T}(L_6, L_7)((b, c), (a, d)), \end{aligned}$$

which shows that  $\text{Lex}_{\tilde{T}, T}(L_6, L_7)$  is not  $T$ -linear.

Note that the condition of involutiveness of  $N_T$  in particular excludes all t-norms without zero divisors. Therefore, lexicographic compositions of non-trivial (i.e. non-crisp) fuzzy orderings with respect to the popular minimum and product t-norms are problematic, if not meaningless. The



reason for this is simple. It is known<sup>11</sup> that the only strict fuzzy ordering included in a fuzzy ordering that is strictly greater than zero (e.g. like  $L_4$  and  $L_5$  from Example 1.2) is the trivial zero relation. When it comes to lexicographic composition, the strict fuzzy ordering induced by the first component relation plays a crucial role. If it vanishes, no meaningful lexicographic composition that preserves linearity properties can be expected. As an example, see Figure 1.4. It shows a cut view of the fuzzy ordering  $\text{Lex}_{T_P, T_P}(L_5, L_4)$ . It is easy to see that  $\text{Lex}_{T_P, T_P}(L_5, L_4)$  is nothing else but the Cartesian product of  $L_5$  and  $L_4$ , which is of course not  $T_P$ -linear.

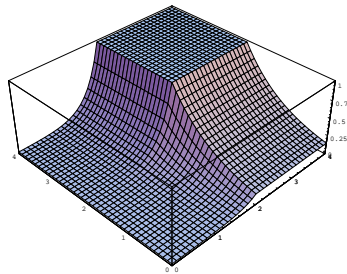


Fig. 1.4. A cut view of  $\text{Lex}_{T_P, T_P}(L_5, L_4)$ .

The final and most important question is whether the lexicographic composition of two  $T$ -linear fuzzy orderings is again  $T$ -linear. Strong completeness always implies  $T$ -linearity,<sup>17</sup> thus, strongly complete fuzzy orderings are a sub-class of  $T$ -linear fuzzy orderings (no matter which  $T$  we choose). If the involutiveness of  $N_T$  is a necessary condition for meaningful results in Proposition 1.3, there is no point in considering a t-norm that does not induce an involutive negation any further.

**Theorem 1.3.** *Let us again make the same assumptions as for Theorem 1.2. If  $L_1$  and  $L_2$  are  $T$ -linear fuzzy orderings and the residual negation  $N_T$  is involutive, then the following fuzzy ordering is  $T$ -linear:*

$$\begin{aligned} \text{Lex}_{T_M, T}(L_1, L_2)((x_1, x_2), (y_1, y_2)) = \\ \max(\min(L_1(x_1, y_1), L_2(x_2, y_2)), \\ \min(L_1(x_1, y_1), N_T(L_1(y_1, x_1)))) \end{aligned}$$

Obviously, Theorem 1.3 does not allow any choice of the aggregating t-norm  $\tilde{T}$  as in the original construction in Theorem 1.2, but enforces the

choice of the minimum t-norm (i.e.  $\tilde{T} = T_M$ ). This is not an arbitrary restriction, but a necessary condition, as the following example demonstrates.

**Example 1.5.** Consider an arbitrary left-continuous t-norm  $T$  that induces an involutive negation  $N_T$  and assume that  $\tilde{T} < T_M$ . Then there exists a  $y \in ]0, 1[$  such that  $\tilde{T}(y, y) < y$ . Now let us consider the following two fuzzy relations:

$$\begin{array}{c|cc} L_8 & a & b \\ \hline a & 1 & y \\ b & 1 & 1 \end{array} \quad \begin{array}{c|cc} L_9 & c & d \\ \hline c & 1 & y \\ d & N_T(y) & 1 \end{array}$$

It is easy to see that  $L_8$  and  $L_9$  are  $T$ -linear fuzzy orderings with respect to  $T$  and some  $T$ -equivalences (the exact definition of them is not important at this point). Now we can compute:

$$\begin{aligned} \text{Lex}_{\tilde{T}, T}(L_8, L_9)((a, c), (b, d)) &= \max(\tilde{T}(y, y), \min(y, N_T(1))) = \tilde{T}(y, y) \\ \text{Lex}_{\tilde{T}, T}(L_8, L_9)((b, d), (a, c)) &= \max(\tilde{T}(1, N_T(y)), \min(1, N_T(y))) = N_T(y) \end{aligned}$$

If  $\text{Lex}_{\tilde{T}, T}(L_8, L_9)$  was linear, the following inequality would be fulfilled:

$$N_T(\text{Lex}_{\tilde{T}, T}(L_8, L_9)((b, d), (a, c))) \leq \text{Lex}_{\tilde{T}, T}(L_8, L_9)((a, c), (b, d))$$

However, we obtain:

$$\begin{aligned} N_T(\text{Lex}_{\tilde{T}, T}(L_8, L_9)((b, d), (a, c))) &= N_T(N_T(y)) \\ &= y > \tilde{T}(y, y) \\ &= \text{Lex}_{\tilde{T}, T}(L_8, L_9)((a, c), (b, d)) \end{aligned}$$

Therefore,  $\text{Lex}_{\tilde{T}, T}(L_8, L_9)$  can never be  $T$ -linear if  $\tilde{T} < T_M$ . This example, therefore, justifies the assumptions of Theorem 1.3.

### 1.5. Conclusion

In this paper, we have introduced an approach to lexicographic composition of similarity-based fuzzy orderings. This construction, in principle, works for all choices of t-norms. However, *the* essential property of lexicographic compositions—that the lexicographic composition of linear orderings is again a linear ordering on the product domain—is only maintained if the underlying t-norm  $T$  induces an involutive negation (in particular, including nilpotent t-norms and the nilpotent minimum). This once more confirms the viewpoint that such t-norms are most adequate choices in fuzzy relations theory, fuzzy preference modeling and related fields.<sup>11,17,20–22</sup>

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