

# Valverde-Style Representation Results in a Graded Framework

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## Abstract

This paper generalizes the well-known representations of fuzzy preorders and similarities according to Valverde to the graded framework of Fuzzy Class Theory (FCT). The results demonstrate that FCT is a powerful tool and that new results and interesting constructions can be obtained by considering fuzzy relations in the graded framework of FCT.

**Keywords:** Fuzzy Class Theory, Fuzzy Relations, Graded Properties, MTL Logic.

## 1 Introduction

This paper aims at generalizing two of the most important and influential theorems in the theory of fuzzy relations—Valverde’s representation theorems for fuzzy preorders and similarities [23]:

Consider a fuzzy relation  $R: U \times U \rightarrow [0, 1]$ .  $R$  is a fuzzy preorder with respect to some left-continuous triangular norm  $*$  if and only if there exists a family  $(f_i)_{i \in I}$  of score functions  $U \rightarrow [0, 1]$  such that  $R$  can be represented as (for all  $x, y \in U$ )

$$R(x, y) = \inf_{i \in I} (f_i(x) \Rightarrow f_i(y)),$$

where  $\Rightarrow$  is the residual implication of  $*$ . Moreover,  $R$  is a similarity with respect to  $*$  if and only if there exists a family  $(f_i)_{i \in I}$  of score functions  $U \rightarrow [0, 1]$  such that  $R$  can be represented as (for all  $x, y \in U$ )

$$R(x, y) = \inf_{i \in I} (f_i(x) \Leftrightarrow f_i(y)),$$

where  $\Leftrightarrow$  is the residual bi-implication of  $*$ . These two results hold for fuzzy preorders and similarities, respectively, but clearly they do not provide us with any insight if the relations fail to fulfill those requirements.

In the early 1990’s, Gottwald has introduced what he calls *graded properties of fuzzy relations* [17–19], a framework in which it is possible to deal with partial—*graded*—fulfillment of properties like reflexivity, transitivity, etc. In this framework, it is not only possible to define properties in a graded way, but also to generalize theorems on fuzzy relations in the sense that an assertion about a certain sub-class of fuzzy relations holds to a degree that depends on the degree to which the relations fulfill the necessary properties. Even though these ideas seem obvious and meaningful, Gottwald’s approach unfortunately found only little resonance (exceptions are, for instance, [6,21]), mainly because it is not a full-fledged axiomatic framework and is not strictly separating syntax from semantics. For this reason, proofs are complicated and difficult.

With the advent of Fuzzy Class Theory (FCT) [3], a formal axiomatic framework is available in which it is just natural to consider properties of fuzzy relations in a graded manner. Notions are inspired by (and derived from) the corresponding notions of classical mathematics [4]; the syntax of FCT is close to the syntax of classical mathematical theories and the proofs in FCT resemble the proofs of the corresponding classical theorems. Therefore, it is technically easier to handle graded properties of fuzzy relations than in Gottwald’s previous works and it is possible to access deeper results than in Gottwald’s framework.

This paper is devoted to this advancement, concentrating on Valverde’s famous representation theorems for fuzzy preorders and similarities. In the tradition of Cantor [9], Valverde uses score functions to represent relations. As these score functions map to the unit interval, they can also be considered as fuzzy sets, which facilitates a reformulation of these results in FCT.

The paper is organized as follows. After some preliminaries concerning FCT in Section 2, we introduce graded properties in Section 3. Section 4 is devoted to the generalization of Valverde’s representation theorem for fuzzy preorders, while Section 5 deals with the

corresponding results for similarities. Note that this paper is an excerpt of a larger manuscript that has been submitted for publication [2]. For background information and proof details, readers are referred to this upcoming article.

## 2 Preliminaries

We aim this paper at researchers in the theory and applications of fuzzy relations to attract their interest in graded theories of fuzzy relations. In the traditional theory of fuzzy relations, it is not usual to separate formal syntax from semantics as it is the case in FCT. So it may be difficult for some readers who are new to FCT to follow the results. Therefore, we would like to provide the readers basically with a dictionary that improves understanding of the results in this paper and that demonstrates how the results would translate to the traditional language of fuzzy relations. For a more formal introduction to FCT, readers are referred to the appendix of this paper and the freely available primer [5].

FCT strictly distinguishes between its syntax and semantics; that is, we distinguish between a formal syntax of formulae and the fuzzy relations modeling them. This feature has two important consequences: (i) To make this distinction clear, the objects of the formal theory are called *fuzzy classes* and not *fuzzy sets*. The name *fuzzy set* is reserved for membership functions in the *models* of the theory (see Appendix). Nevertheless, the theorems of FCT about fuzzy classes are always valid for fuzzy *sets* and fuzzy relations. Thus, whenever we speak of classes, the reader can always safely substitute usual fuzzy sets for “classes”. (ii) FCT screens off direct references to truth values; truth degrees belong to the *semantics* of FCT, rather than to its syntax. Thus, there are *no variables for truth degrees* in the language of FCT. The degree to which an element  $x$  belongs to a fuzzy class  $A$  is expressed simply by the atomic formula  $x \in A$  (which can alternatively be written in a more traditional way as  $Ax$ ).

The algebraic structure of truth degrees in the semantics of FCT is that of  $\text{MTL}_{\Delta}$ -chains [13,20].<sup>1</sup> If the domain of truth values is the unit interval  $[0, 1]$ ,  $\text{MTL}_{\Delta}$ -chains are characterized as algebras

$$([0, 1], *, \Rightarrow, \min, \max, 0, 1, \Delta),$$

where  $*$  is a left-continuous t-norm,  $\Rightarrow$  is its residual implication, and  $\Delta$  is a unary operation defined as

$$\Delta x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

<sup>1</sup>Note that even though in the first paper [3] FCT was based on the logic LII [14], the logic  $\text{MTL}_{\Delta}$  is sufficient for our present needs.

This means that we can translate the results to the language of fuzzy relations in the following way, where we may specify an arbitrary universe of discourse  $U$  and a left-continuous t-norm  $*$  (with the residuum  $\Rightarrow$ ):

FCT	Fuzzy relations
object variable $x$	element $x \in U$
(fuzzy) class $A$	fuzzy set $A \in \mathcal{F}(U)$
2nd-order fuzzy class $\mathcal{A}$	fuzzy set $\mathcal{A} \in \mathcal{F}(\mathcal{F}(U))$
unary predicate	fuzzy subset of $U$ , $\mathcal{F}(U)$ , etc.
$n$ -ary predicate	$n$ -ary f. rel. on $U^n$ , $(\mathcal{F}(U))^n$ , etc.
strong conjunction $\&$	left-continuous t-norm $*$
implication $\rightarrow$	residual implication $\Rightarrow$
weak conjunction $\wedge$	minimum
weak disjunction $\vee$	maximum
negation $\neg$	the function $\neg x = (x \Rightarrow 0)$
equivalence $\leftrightarrow$	bi-residuum: $\min(x \Rightarrow y, y \Rightarrow x)$
universal quantifier $\forall$	infimum
existential quantifier $\exists$	supremum
predicate $=$	crisp identity
predicate $\in$	evaluation of membership function
class term $\{x \mid \varphi(x)\}$	f. set def. as $Ax = \varphi(x)$ , for $x \in U$

For details on the syntax of FCT and defined notions see Appendix A and [3,5].

Let us now shortly consider two examples. For instance, the truth degree of  $A \subseteq B$ , defined by the formula  $(\forall x)(x \in A \rightarrow x \in B)$  in FCT (see Definition A.3) is in an  $\text{MTL}_{\Delta}$ -chain computed as

$$\inf_{x \in U} (Ax \Rightarrow Bx),$$

which is a well-known concept of fuzzy inclusion (see [1,6,7,18] and many more). The degree of reflexivity  $\text{Refl}(R)$ , defined as  $(\forall x)Rxx$  in Section 3, is nothing else but

$$\inf_{x \in U} Rxx.$$

We shall now proceed to how the theorems in the following sections should be read in a graded way (although they do not necessarily look graded at first glance). In traditional (fuzzy) logic, a theorem is read as follows:

*If* a (non-graded) assumption is true  
(i.e., fully true, since non-graded),  
*then* a (non-graded) conclusion is (fully) true.

If an implication is provable in FCT, by soundness, it always holds to degree 1. Now take into account that, in all  $\text{MTL}_{\Delta}$ -chains (comprising all standard  $\text{MTL}_{\Delta}$ -chains), the following correspondence holds:

$$(x \Rightarrow y) = 1 \text{ if and only if } x \leq y.$$

So an implication that we can prove in FCT can be read as follows:

*The more* a (graded) assumption is true  
(even if partially),  
*the more* a (graded) conclusion is true  
(i.e., at least as true as the assumption).

In other words, the truth degree of an assumption is a lower bound for the truth degree of the conclusion in provable implications.

**Remark:** To motivate and illustrate the results in this paper, we will use several examples. In order to make them compact and readable, we will, *in examples*, deviate from our principle to keep formulae separate from their semantics. Instead of mentioning models over some logics, we will simply say that we use some standard logic, for instance, standard Łukasiewicz logic (standing for the standard  $\text{MTL}_\Delta$ -chain induced by the Łukasiewicz t-norm; analogously for other logics). In examples, we shall furthermore not distinguish between predicate symbols and the fuzzy sets or relations that model them. Instead of saying that a certain model of a fuzzy predicate  $R$  fulfills reflexivity to a degree of 0.8, we will simply write  $\text{Refl}(R) = 0.8$ . This is not the cleanest way of writing it, but it is short and expressive, and it should always be clear to the reader what is meant.

### 3 Basic Graded Properties of Fuzzy Relations

As an important prerequisite, we first define graded variants of well-known properties of fuzzy relations in the framework of FCT.

**Definition 3.1** In FCT, we define basic properties of fuzzy relations as follows:

$$\begin{aligned} \text{Refl}(R) &\equiv_{\text{df}} (\forall x)Rxx \\ \text{Sym}(R) &\equiv_{\text{df}} (\forall x, y)(Rxy \rightarrow Ryx) \\ \text{Trans}(R) &\equiv_{\text{df}} (\forall x, y, z)(Rxy \& Ryz \rightarrow Rxz) \end{aligned}$$

**Example 3.2** Let us shortly provide a simple example to illustrate the concepts introduced in Definition 3.1. Consider the domain  $U = \{1, \dots, 6\}$  and the following fuzzy relation (for convenience, in matrix notation):

$$P_1 = \begin{pmatrix} 1.0 & 1.0 & 0.5 & 0.4 & 0.3 & 0.0 \\ 0.8 & 1.0 & 0.4 & 0.4 & 0.3 & 0.0 \\ 0.7 & 0.9 & 1.0 & 0.8 & 0.7 & 0.4 \\ 0.9 & 1.0 & 0.7 & 1.0 & 0.9 & 0.6 \\ 0.6 & 0.8 & 0.8 & 0.7 & 1.0 & 0.7 \\ 0.3 & 0.5 & 0.6 & 0.4 & 0.7 & 1.0 \end{pmatrix}$$

It is easy to check that  $P_1$  is a fuzzy preorder with respect to the Łukasiewicz t-norm  $\max(x + y - 1, 0)$ , hence, taking standard Łukasiewicz logic, we obtain  $\text{Refl}(P_1) = 1$  and  $\text{Trans}(P_1) = 1$ . In this setting, one can easily compute  $\text{Sym}(P_1) = 0.4$  (note that, for a finite fuzzy relation  $R$ , in standard Łukasiewicz logic,  $\neg \text{Sym}(R)$  is nothing else but the largest difference between the two values  $Rxy$  and  $Ryx$ ).

Now let us see what happens if we add some disturbances to  $P_1$ . Consider the following fuzzy relation:

$$P_2 = \begin{pmatrix} 1.00 & 1.00 & 0.56 & 0.40 & 0.30 & 0.00 \\ 0.87 & 1.00 & 0.33 & 0.44 & 0.26 & 0.02 \\ 0.67 & 0.92 & 0.93 & 0.87 & 0.70 & 0.39 \\ 0.93 & 1.00 & 0.64 & 1.00 & 0.97 & 0.67 \\ 0.52 & 0.79 & 0.82 & 0.71 & 1.00 & 0.59 \\ 0.27 & 0.50 & 0.61 & 0.41 & 0.72 & 1.00 \end{pmatrix}$$

Simple computations give the following results:  $\text{Refl}(P_2) = 0.93$ ,  $\text{Sym}(P_2) = 0.41$ ,  $\text{Trans}(P_2) = 0.85$  (for standard Łukasiewicz logic again).

**Example 3.3** Consider  $U = \mathbb{R}$  and let us define the following parameterized class of fuzzy relations (with  $a, c > 0$ ):

$$E_{a,c}xy = \min(1, \max(0, a - \frac{1}{c}|x - y|))$$

It is well known that, for  $a = 1$ , we obtain fuzzy equivalence relations with respect to the Łukasiewicz t-norm [10, 11, 22, 23]; hence, using standard Łukasiewicz logic again,  $\text{Refl}(E_{1,c}) = 1$ ,  $\text{Sym}(E_{1,c}) = 1$ , and  $\text{Trans}(E_{1,c}) = 1$  for all  $c > 0$ . On the contrary, it is obvious that reflexivity in the non-graded manner cannot be maintained for  $a < 1$ . Actually, we obtain

$$\text{Refl}(E_{a,c}) = \min(1, a).$$

for all  $a, c > 0$ . Similarly, it is a well-known fact that, for  $a > 1$ , transitivity in the non-graded sense is violated [12]. Regarding graded transitivity, we obtain the following:

$$\text{Trans}(E_{a,c}) = \min(1, \max(0, 2 - a))$$

None of these results depends on the parameter  $c$ , as  $c$  only corresponds to a re-scaling of the domain. We can conclude that the larger  $a$ , the more reflexive, but less transitive,  $E_{a,c}$  is. Figure 1 shows two examples.

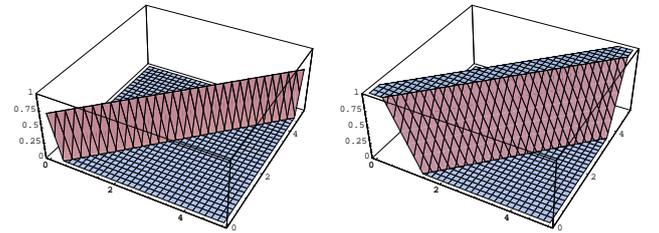


Figure 1: The fuzzy relations  $E_{0.7,2}$  (left) and  $E_{1.4,1}$  (right). From Example 3.3, we can infer that  $\text{Refl}(E_{0.7,2}) = 0.7$ ,  $\text{Trans}(E_{0.7,2}) = \text{Refl}(E_{1.4,1}) = 1$ , and  $\text{Trans}(E_{1.4,1}) = 0.6$ .

In classical mathematics, special properties of relations are rarely studied completely independently of

each other. Instead, these properties most often occur in some combinations in the definitions of special classes of relations—with (pre)orders and equivalence relations being two most fundamental examples. The same is true in the theory of fuzzy relations, where fuzzy (pre)orders and fuzzy equivalence relations are the most important classes. Compound properties of this kind are defined as conjunctions of some of the simple properties of Definition 3.1. In the non-graded case, the properties are crisp, so the conjunction we need is the classical Boolean conjunction. In FCT, however, all properties are graded, so it indeed matters which conjunction we take. Thus, besides the (more usual) combinations by strong conjunction  $\&$  (corresponding to the t-norm in the standard case), we also define their weak variants combined by weak conjunction (corresponding to the minimum). In this paper, we restrict to the investigation of fuzzy preorders and similarities.<sup>2</sup>

**Definition 3.4** In FCT, we define the following compound properties of fuzzy relations:

$$\begin{aligned} \text{Preord}(R) &\equiv_{\text{df}} \text{Refl}(R) \ \& \ \text{Trans}(R) \\ \text{wPreord}(R) &\equiv_{\text{df}} \text{Refl}(R) \ \wedge \ \text{Trans}(R) \\ \text{Sim}(R) &\equiv_{\text{df}} \text{Refl}(R) \ \& \ \text{Sym}(R) \ \& \ \text{Trans}(R) \\ \text{wSim}(R) &\equiv_{\text{df}} \text{Refl}(R) \ \wedge \ \text{Sym}(R) \ \wedge \ \text{Trans}(R) \end{aligned}$$

**Example 3.5** Let us shortly revisit Example 3.2. We can conclude the following:

$$\begin{array}{ll} \text{Preord}(P_1) = 1 & \text{Preord}(P_2) = 0.78 \\ \text{wPreord}(P_1) = 1 & \text{wPreord}(P_2) = 0.85 \\ \text{Sim}(P_1) = 0.4 & \text{Sim}(P_2) = 0.19 \\ \text{wSim}(P_1) = 0.4 & \text{wSim}(P_2) = 0.41 \end{array}$$

The values in the second column once more demonstrate why it is justified to speak of strong and weak properties—the properties with strong conjunction get smaller truth degrees and thus are harder to fulfil. Obviously, the implications  $\text{Preord}(R) \rightarrow \text{wPreord}(R)$  and  $\text{Sim}(R) \rightarrow \text{wSim}(R)$  hold.

**Example 3.6** For the family of fuzzy relations defined in Example 3.3, we obtain the interesting result

$$\text{Preord}(E_{a,c}) = \text{wPreord}(E_{a,c}) = \max(0, 1 - |1 - a|),$$

from which we can infer that  $\text{Preord}(E_{a,c}) = \text{wPreord}(E_{a,c}) = 1$  if and only if  $a = 1$ . Note that  $\text{Sym}(E_{a,c}) = 1$ , so  $\text{Sim}(E_{a,c}) = \text{Preord}(E_{a,c})$  and  $\text{wSim}(E_{a,c}) = \text{wPreord}(E_{a,c})$ , which implies that  $\text{Sim}(E_{a,c}) = \text{wSim}(E_{a,c}) = 1$  if and only if  $a = 1$ .

<sup>2</sup>In line with Zadeh’s original work [25], we use the term *similarity (relation)* synonymously for fuzzy equivalence (relation).

## 4 Representation of Fuzzy Preorders

This section aims at generalizing Valverde’s representation theorem for fuzzy preorders. We will proceed as follows: we first generalize Fodor’s characterization by means of traces and then use this characterization to prove the generalization of Valverde’s theorem. Note that Valverde’s original proof [23] implicitly follows the same lines.

So given a fuzzy relation  $R$ , let us first consider the fuzzy relation  $R^\ell$  defined as

$$R^\ell xy \equiv_{\text{df}} (\forall z)(Rzx \rightarrow Rzy)$$

This is called the *left trace* of  $R$  [15, 16]. Analogously, we can define the *right trace* as

$$R^r xy \equiv_{\text{df}} (\forall z)(Ryz \rightarrow Rxz).$$

Observe the meaning of the following expressions:

$$\begin{aligned} R^\ell \subseteq R &\leftrightarrow (\forall x, y)[(\forall z)(Rzx \rightarrow Rzy) \rightarrow Rxy] \\ R \subseteq R^\ell &\leftrightarrow (\forall x, y)[Rxy \rightarrow (\forall z)(Rzx \rightarrow Rzy)] \\ R \approx R^\ell &\leftrightarrow (\forall x, y)[Rxy \leftrightarrow (\forall z)(Rzx \rightarrow Rzy)] \end{aligned}$$

Now we can formulate characterizations of graded reflexivity and transitivity, which are not difficult to prove in FCT.

**Theorem 4.1** *The following properties hold in FCT:*

$$\begin{aligned} \text{Refl}(R) &\leftrightarrow R^\ell \subseteq R \\ \text{Trans}(R) &\leftrightarrow R \subseteq R^\ell \end{aligned}$$

As a corollary we obtain graded versions of Fodor’s characterizations [15, Theorems 4.1, 4.3, and Corollary 4.4]. For the two notions of fuzzy equality  $\approx$  and  $\cong$ , see Definition A.3.

**Corollary 4.2** *The following is provable in FCT:*

$$\begin{aligned} \text{wPreord}(R) &\leftrightarrow R \approx R^\ell \\ \text{Preord}(R) &\leftrightarrow R \cong R^\ell \\ R \approx^2 R^\ell &\longrightarrow \text{Preord}(R) \longrightarrow R \approx R^\ell \end{aligned}$$

Note that, regardless of the symmetry of  $R$ , we can replace  $R^\ell$  in the above characterizations by the right trace as well.

Now we have all prerequisites for formulating and proving a graded version of Valverde’s representation theorem for preorders. In order to make notations more compact, let us define two graded notions of *Valverde preorder representation* (a strong one and a

weak one), for a given fuzzy relation  $R$  and a fuzzy class of fuzzy classes  $\mathcal{A}$ :

$$\begin{aligned}\text{ValP}(R, \mathcal{A}) &\equiv_{\text{df}} R \approx \{\langle x, y \rangle \mid (\forall A \in \mathcal{A})(Ax \rightarrow Ay)\} \\ \text{wValP}(R, \mathcal{A}) &\equiv_{\text{df}} R \approx \{\langle x, y \rangle \mid (\forall A \in \mathcal{A})(Ax \rightarrow Ay)\}\end{aligned}$$

The predicates ValP and wValP express the degree to which the fuzzy class  $\mathcal{A}$  represents the relation  $R$ .

Then we can prove the following essential result for preorders and weak preorders.

**Theorem 4.3** FCT *proves the following:*

$$\begin{aligned}(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \ \&\ \text{ValP}^2(R, \mathcal{A}) \longrightarrow \text{Preord}(R) \\ &\longrightarrow (\exists \mathcal{A})(\text{Crisp}(\mathcal{A}) \ \&\ \text{ValP}(R, \mathcal{A})) \\ (\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \ \&\ \text{wValP}^3(R, \mathcal{A}) \longrightarrow \text{wPreord}(R) \\ &\longrightarrow (\exists \mathcal{A})(\text{Crisp}(\mathcal{A}) \ \&\ \text{wValP}(R, \mathcal{A}))\end{aligned}$$

It can be shown that the exponents in this theorem cannot be lowered, see [2] for a counterexample. Obviously, this theorem is more complicated than Valverde's original result; it is an example where the graded framework does not provide us with just a plain copy of the non-graded (or crisp) result. The following corollary gives us a result that is comparable with Valverde's original theorem.

**Corollary 4.4** *The following equivalences are provable in FCT:*

$$\begin{aligned}\Delta \text{Preord}(R) &\longleftrightarrow R = R^\ell \longleftrightarrow & (1) \\ (\exists \mathcal{A})(\Delta(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \ \&\ \Delta \text{ValP}(R, \mathcal{A})) & & (2) \\ \longleftrightarrow (\exists \mathcal{A})(\text{Crisp}(\mathcal{A}) \ \&\ \Delta \text{ValP}(R, \mathcal{A})) & & (3)\end{aligned}$$

Observe that also the analogous formulas with wPreord and wValP are equivalent to those in this corollary.

**Example 4.5** Let us shortly revisit Example 3.2 (in which we use standard Łukasiewicz logic). The fuzzy relation  $P_1$  was actually constructed from the following crisp family of three fuzzy sets  $\mathcal{A} = \{A_1, A_2, A_3\}$  that are defined as follows (for convenience, in vector notation):

$$\begin{aligned}A_1 &= (0.7, 0.8, 0.2, 0.5, 0.4, 0.6) \\ A_2 &= (0.3, 0.5, 0.6, 0.4, 0.7, 1.0) \\ A_3 &= (1.0, 1.0, 0.6, 0.4, 0.3, 0.0)\end{aligned}$$

Although the formula (3) is a perfect copy of Valverde's non-graded representation, the corollary still has graded elements—note that in (2), the class  $\mathcal{A}$  may still be a fuzzy class of fuzzy classes, if only it satisfies  $\Delta(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$ . Recall that in Gödel logic, this

condition is fulfilled by *all* fuzzy classes  $\mathcal{A}$ , and that in any logic it is satisfied by a system  $\mathcal{A}$  in a model if all degrees of membership in  $\mathcal{A}$  are idempotent with respect to conjunction.

The degree of  $A \in \mathcal{A}$  may be considered as a weighting factor that controls the influence of a specific  $A$  on the final result. Corollary 4.4 requires all membership degrees in  $\mathcal{A}$  to be idempotent to ensure that the relation represented by  $\mathcal{A}$  is a fuzzy preorder, but its graded version in Theorem 4.3 also shows that (loosely speaking) it will *almost* be a fuzzy preorder if  $\mathcal{A}$  *almost* satisfies  $\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}$  (e.g., in standard Łukasiewicz logic if it is close to crispness).

**Example 4.6** Let us consider a  $[0, 1]$ -valued fuzzy logic with the triangular norm

$$x * y = \begin{cases} \max(x + y - \frac{1}{2}, 0) & \text{if } x \in [0, \frac{1}{2}]^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

i.e. a simple ordinal sum with a scaled Łukasiewicz t-norm in  $[0, \frac{1}{2}]^2$  and the Gödel t-norm anywhere else. It is clear that the set of idempotent elements of this t-norm is  $\{0\} \cup [\frac{1}{2}, 1]$  and that the corresponding residual implication is given as

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ \max(y, \frac{1}{2} - x + y) & \text{otherwise.} \end{cases}$$

Now reconsider  $U = \{1, \dots, 6\}$  and the three fuzzy sets  $A_1, A_2$  and  $A_3$  from Example 4.5 and define a fuzzy class of fuzzy classes  $\mathcal{A}$  such that  $\mathcal{A}A_1 = 0.9$ ,  $\mathcal{A}A_2 = 1.0$ , and  $\mathcal{A}A_3 = 0.8$ . Since all three values are idempotent elements of  $*$ , we can be sure by Theorem 4.3 that the construction  $R_1 =_{\text{df}} \{\langle x, y \rangle \mid (\forall A \in \mathcal{A})(Ax \rightarrow Ay)\}$  always gives us a fuzzy preorder in the given logic. In this particular example, we obtain the following:

$$R_1 = \begin{pmatrix} 1.0 & 1.0 & 0.2 & 0.4 & 0.3 & 0.0 \\ 0.3 & 1.0 & 0.2 & 0.4 & 0.3 & 0.0 \\ 0.3 & 0.5 & 1.0 & 0.4 & 0.3 & 0.0 \\ 0.4 & 1.0 & 0.2 & 1.0 & 0.4 & 0.1 \\ 0.3 & 0.5 & 0.3 & 0.4 & 1.0 & 0.2 \\ 0.3 & 0.5 & 0.2 & 0.4 & 0.4 & 1.0 \end{pmatrix}$$

If we repeat this construction and define a fuzzy relation  $R_2 =_{\text{df}} \{\langle x, y \rangle \mid (\forall A \in \mathcal{A})(Ax \rightarrow Ay)\}$  with  $\mathcal{A}$  defined as above, but the connectives interpreted in standard Łukasiewicz logic, we obtain the following:

$$R_2 = \begin{pmatrix} 1.0 & 1.0 & 0.6 & 0.6 & 0.5 & 0.2 \\ 0.8 & 1.0 & 0.5 & 0.6 & 0.5 & 0.2 \\ 0.7 & 0.9 & 1.0 & 0.8 & 0.9 & 0.6 \\ 0.9 & 1.0 & 0.8 & 1.0 & 1.0 & 0.8 \\ 0.6 & 0.8 & 0.9 & 0.7 & 1.0 & 0.9 \\ 0.3 & 0.5 & 0.6 & 0.4 & 0.7 & 1.0 \end{pmatrix}.$$

Straightforward calculations show that  $\text{Refl}(R_2) = 1$  and  $\text{Trans}(R_2) = \text{Preord}(R_2) = \text{wPreord}(R_2) = 0.8$ . This is not at all contradicting to Theorem 4.3, as  $\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}$  holds only to a degree of 0.6 in standard Lukasiewicz logic.

## 5 Representations of Similarities

In his landmark paper [23], Valverde not only considers fuzzy preorders, but also similarities (as obvious from the title of this paper). So the question naturally arises how we can modify the above results in the presence of symmetry. As will be seen next, the modifications are not as straightforward as in the non-graded case. We first define a fuzzy relation  $R^{\ell s}$  as

$$R^{\ell s}xy \equiv_{\text{df}} (\forall z)(Rzx \leftrightarrow Rzy)$$

(for a given fuzzy relation  $R$ ). There is no particular name for this fuzzy relation in literature. In analogy to Section 4, let us call it *left symmetric trace* of  $R$ .

The following lemma demonstrates how this notion is related to the defining properties of similarities.

**Theorem 5.1** *The following are theorems of FCT:*

$$R^{\ell s} \subseteq R \leftrightarrow \text{Refl}(R) \quad (4)$$

$$R \subseteq R^{\ell s} \rightarrow \text{Trans}(R) \quad (5)$$

$$R \cong R^{\ell s} \rightarrow \text{Sym}(R) \quad (6)$$

$$\text{Sym}(R) \& \text{Trans}(R) \rightarrow R \subseteq R^{\ell s} \quad (7)$$

The following theorem provides us with an analogue of Corollary 4.2, unfortunately, with looser bounds on the left-hand side.

**Corollary 5.2** *FCT proves:*

$$R \approx^4 R^{\ell s} \longrightarrow R \cong^2 R^{\ell s} \longrightarrow \text{Sim}(R)$$

$$\text{Sim}(R) \longrightarrow R \cong R^{\ell s} \longrightarrow R \approx R^{\ell s}$$

$$R \approx^2 R^{\ell s} \longrightarrow R \cong R^{\ell s} \longrightarrow \text{wSim}(R)$$

$$\text{wSim}^2(R) \longrightarrow R \approx R^{\ell s}$$

The question arises whether it is really necessary to require  $\cong$  rather than  $\approx$  in (6). The following example tells us that this is indeed the case. It also implies that  $R \approx R^{\ell s} \rightarrow \text{wSim}(R)$  does *not* hold in general.

**Example 5.3** Consider  $U = \{1, 2\}$ , standard Lukasiewicz logic, and the following fuzzy relation:

$$R = \begin{pmatrix} 0.5 & 1.0 \\ 0.0 & 0.5 \end{pmatrix}$$

It is obvious that  $\text{Refl}(R) = 0.5$  and  $\text{Sym}(R) = 0$ . Moreover, routine calculations show  $\text{Trans}(R) = 1$  and that  $R^{\ell s}$  is given as follows:

$$R^{\ell s} = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$$

So, we finally obtain that  $(R \approx R^{\ell s}) = 0.5$ , while  $(R \cong R^{\ell s}) = 0$ .

Now we can formulate a graded version of Valverde's representation theorem for similarities. Analogously to the above considerations, let us define the graded notion of *Valverde similarity representation* (strong one and weak one) for a given fuzzy relation  $R$  and a fuzzy class  $\mathcal{A}$ :

$$\text{ValS}(R, \mathcal{A}) \equiv_{\text{df}} R \cong \{ \langle x, y \rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay) \}$$

$$\text{wValS}(R, \mathcal{A}) \equiv_{\text{df}} R \approx \{ \langle x, y \rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay) \}$$

In the same way as for preorders, we can prove Valverde's representation theorem of similarities and weak similarities.

**Theorem 5.4** *FCT proves the following:*

$$(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \& \text{ValS}^3(R, \mathcal{A}) \rightarrow \text{Sim}(R) \quad (8)$$

$$\text{Sim}(R) \rightarrow (\exists \mathcal{A})(\text{Crisp}(\mathcal{A}) \& \text{ValS}(R, \mathcal{A})) \quad (9)$$

$$(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \& \text{wValS}^3(R, \mathcal{A}) \rightarrow \text{wSim}(R) \quad (10)$$

$$\text{wSim}^2(R) \rightarrow (\exists \mathcal{A})(\text{Crisp}(\mathcal{A}) \& \text{wValS}(R, \mathcal{A})) \quad (11)$$

Again, this theorem is more complicated than Valverde's original representation of similarities. In the following corollary, analogously to preorders, we can infer a result very similar to Valverde's original theorem in case that the corresponding properties are fulfilled to degree 1.

**Corollary 5.5** *The following equivalences are provable in FCT:*

$$\Delta \text{Sim}(R) \longleftrightarrow R = R^{\ell s} \longleftrightarrow \quad (12)$$

$$(\exists \mathcal{A})(\Delta(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \& \Delta \text{ValS}(R, \mathcal{A})) \quad (13)$$

$$\longleftrightarrow (\exists \mathcal{A})(\text{Crisp}(\mathcal{A}) \& \Delta \text{ValS}(R, \mathcal{A})) \quad (14)$$

Again, like in the case of preorders, we can add equivalent lines with  $\text{wValS}$  instead of  $\text{ValS}$ , and (13) has a graded ingredient—the class  $\mathcal{A}$  may be a fuzzy class of fuzzy classes.

## 6 Concluding Remarks

The present paper has generalized Valverde's famous representation theorems for fuzzy preorders and similarities to the fully graded framework of Fuzzy Class



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